Every subcubic graph is packing $(1, 1, 2, 2, 3)$-colorable

Xujun Liu

Xi'an Jiaotong-Liverpool University

Based on a joint work with Xin Zhang and Yanting Zhang

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Outline

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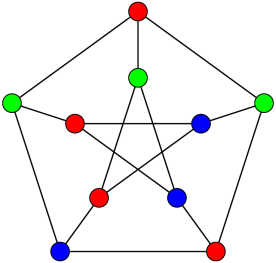

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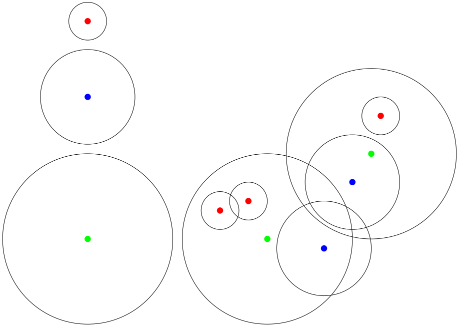

Introduction to packing chromatic number

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For a positive integer i, a set S of vertices in a graph G is **i-independent** if the distance in G between any two distinct vertices of S is at least $i + 1$.

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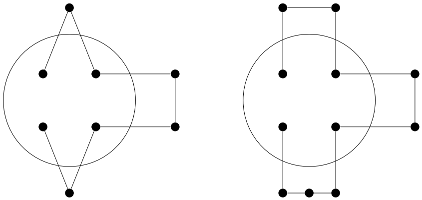

Figure: 1-independent and 2-independent set.

A **proper k-coloring** of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for each $1 \leq i \leq k$, the set V_i is independent.

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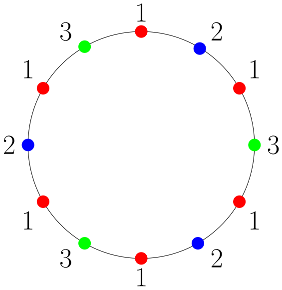

Figure: Upper bound.

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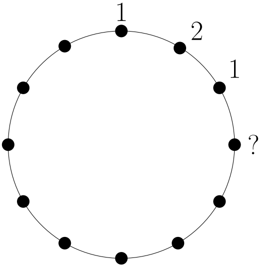

Figure: Lower bound.

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Conjecture (Brešar, Klavžar, Rall, and Wash, 2017)

If G is a subcubic graph, then $\chi_p(D(G)) \leq 5$.

Introduction to packing chromatic number

There are cubic graphs with arbitrarily large packing chromatic number. Furthermore, 'many' cubic graphs have 'high' packing chromatic number:

Theorem (Balogh, Kostochka, L., 2018)

For each fixed integer $k \geq 12$ and $g \geq 2k + 2$, almost every n-vertex cubic graph of girth at least g has the packing chromatic number greater than k.

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In contrast, $\chi_p(D(G))$ is bounded for subcubic graphs G: $\chi_p(D(G))$ is bounded by 8 in this class.

Theorem (Balogh, Kostochka, L., 2019)

For every connected subcubic graph G, graph $D(G)$ has a packing 8-coloring such that color 8 is used at most once.

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• Mortada and Togni showed for subcubic 3-saturated graph that has no adjacent heavy vertices (whose neighbours are all 3-vertices).

Outline

Our main result

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Every subcubic graph G has a packing $(1, 1, 2, 2, 3)$*-coloring.*

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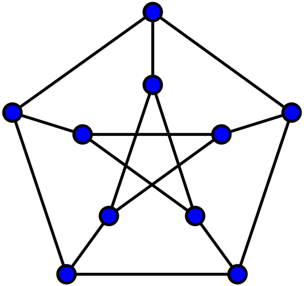

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Let G be a graph and $(s_1, \ldots, s_k)$ be a sequence of non-decreasing positive integers. If G is packing $(s_1, \ldots, s_k)$-colorable, then $D(G)$ is packing $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-colorable.

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Corollary (L., Zhang, Zhang, 2024+)

The 1-subdivision of a subcubic graph G has packing chromatic number at most 6.

Proof Sketch

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Define the graph H_{I_1,I_2} to be the graph H with $V(H) = V(G) - I_1 - I_2$ and $E(H) = \{v_1 v_2 \mid d_G(v_1, v_2) \leq 2, v_1, v_2 \in V(H)\}$.

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We are able to show each component of H is either a tree, an even cycle, or an odd cycle. We only need to analyse the case of an odd cycle to avoid "3-3 conflicts".

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At most one P_2 can be included in a connected component of H.

- Two red P_2's cannot be connected by a black vertex.

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- Configuration C_1 must be alone. There is no configuration C_2.

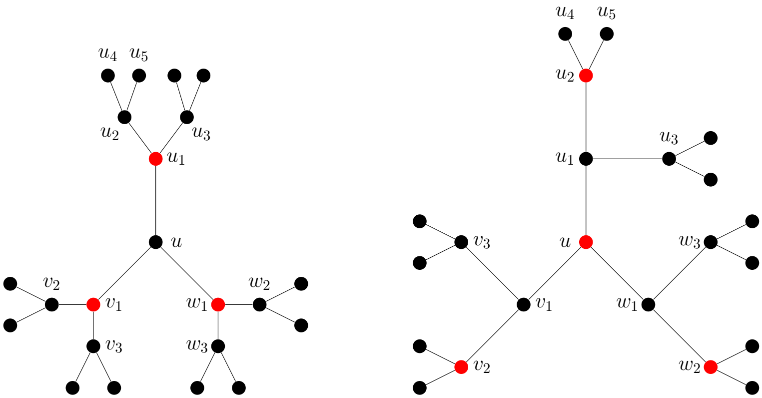

Figure: Configurations C_1 and C_2.

Sketch of Proof

Lemma

$\Delta(H) \leq 3$. *Only one case for* $\Delta(H) = 3$.

• (1) Tree; (2) Even cycle; (3) Odd cycle; (4)Even cycle plus a leaf; (5) Odd cycle plus a leaf.

• We show (4) and (5) are not possible.

Lemma

Each component of H *is not isomorphic to a cycle with one vertex adjacent to a leaf.*

Proof Sketch

- Only need to consider the 3-3 conflicts on two cases of (3).

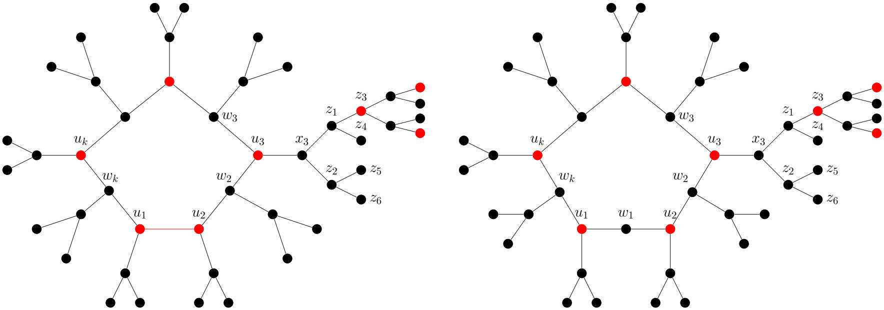

Figure: Cases 1 and 2.

Future Work

Conjecture

Every subcubic graph except the Petersen graph is packing $(1,1,2,2)$*-colorable.*

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Review for edge coloring

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For a simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

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- First generalized by Gastineau and Togni from its vertex counterpart.

Introduction to packing edge coloring

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Given a non-decreasing sequence $S = (s_1, ..., s_k)$ of positive integers, an S-**packing edge-coloring** of a graph G is a decomposition of the edge set of G into disjoint sets $E_1, \ldots, E_k$ such that for each $1 \leq i \leq k$ the distance between any two distinct edges $e_1, e_2 \in E_i$ is at least $s_i + 1$.

- First generalized by Gastineau and Togni from its vertex counterpart.

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A subcubic graph G with $\Delta(G) = 3$ either has a $(1, 1, 1)$-packing edge-coloring or a $(1, 1, 1, 1)$-packing ((1^4)-packing) edge-coloring.

Previous results on strong edge coloring

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Conjecture (Erdős and Nešetřil, 1989)

The upper bound for the strong chromatic index of a graph with maximum degree Δ is $\frac{5}{4}\Delta^2$ when Δ is even and $\frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}$ when Δ is odd.

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• Hurley, de Joannis de Verclos, Kang, 2021: $\chi'_s(G) \leq 1.772\Delta^2$.

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Corollary

Every subcubic graph has a $(1, 2^9)$*-packing edge-coloring.*

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Question (Gastineau and Togni, 2019)

Is it true that all cubic graphs are $(1, 2^7)$*-packing edge-colorable?*

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Conjecture (Gastineau and Togni, 2019; Hocquard et al, 2022)

Every subcubic graph G *admits a* $(1, 1, 2^4)$*-packing edge-coloring.*

Our main results - result 1

Theorem (L., Santana, Short, 2023)

Every subcubic multigraph has a $(1, 2^7)$-packing edge-coloring.

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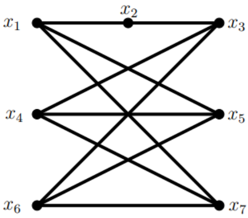

Figure: Sharpness example for our result.

More on the sharpness example

- The largest size of a matching in the graph is three.
- The maximum size of an induced matching is one.
- There are 10 edges and therefore no $(1, 2^6)$-packing edge-coloring and no $(1, 1, 2^3)$-packing edge-coloring.

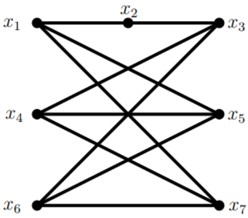

Figure: Non-$(1, 2^6)$-packing and non-$(1, 1, 2^3)$-packing edge-colorable.

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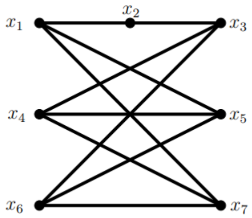

Figure: Non-$(1, 2^6)$-packing and non-$(1, 1, 2^3)$-packing edge-colorable.

- We can find a $(1, 2^7)$-packing and a $(1, 1, 2^4)$-packing edge-coloring.

Remark on the sharpness example

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2. It is also the sharpness example for the Erdős and Nešetřil Conjecture when $\Delta = 3$.

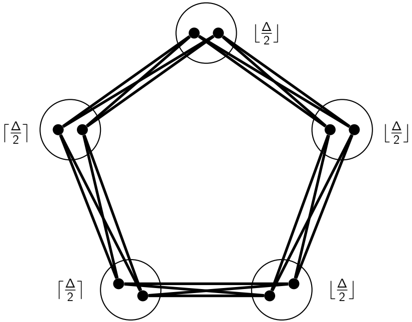

Figure: The sharpness example for the Erdős and Nešetřil Conjecture.

Our main results - result 2

Theorem (L. and Yu, 2024+)

Every connected subcubic graph (with more than 70 vertices) is $(1, 1, 2^4)$*-packing edge-colorable.*

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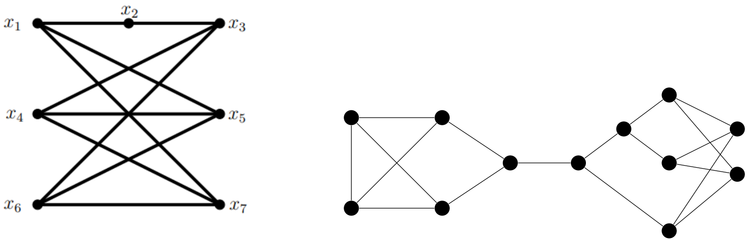

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• Three "2"s trade for one "1" for subcubic graphs.

Future work

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Thank you for listening!